Analytical Modeling of Membrane Bending in Range of Large Deflections

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Abstract – **In early stage of the design process, modeling is a crucial phase as it allows the designer estimating the device performance. Very often, this phase is time-consuming, especially when FEM simulator is used. Hence, a fast and reliable method is desirable. Moreover, in case of simple structures which behavior is described with quite simple equations, the use of analytical model is obvious. In this paper we present a complete model of a membrane bending in range of large deflections that has been developed using FEM simulations.**

Keywords – **MEMS, modeling, analytical model, membrane, large deflection.**

I. INTRODUCTION

Membrane is one of the most common structure used in MEMS devices. It is used in variety of pressure sensors, CMUT, micropumps, etc. In order to design a device with specified performance one need to perform simulation process. In many cases FEM simulation is used. However, this method needs to create a 3-D model and to solve large number of differential equations (depending on mesh density). If the optimization process is needed, the FEM simulations can take a long time disproportionately to structure complexity. Thus, the analytical modeling is a very convenient method to obtain rather precise results in a very short time. The modeling of membrane commonly uses the theory of plates and shells [1] that describes the membrane behavior by one differential equation. The drawback of this method is the function needed to solve the equation. In general, the function approximates the membrane form and its highest order to increase precision makes the solution more complex. It was found [2] that the membrane deflection has linear dependence on applied load in range of small deflection. Thus, the maximal deflection in the membrane center can be described with one simple linear equation that simplifies the solution significantly. The linear range of membrane deflection is commonly used in variety of MEMS due to simplicity of conversion into usable output signal. However, there are some other applications that need larger deflection where small deflection theory become useless. One of them is bulge test technique to characterize materials properties like Young's modulus and residual stress [3]. Then, the large deflection theory has to be used and the equation describing the membrane deflection is third order. This paper presents the complete analytical model of membrane bending taking into consideration large deflection of the membrane. Isotropic materials and commonly used anisotropic silicon are taken into account.

II. MEMBRANE BENDING

We consider a structure that consist of a perfectly clamped membrane of length *a*, width *b* and thickness *h* as shown in figure 1:

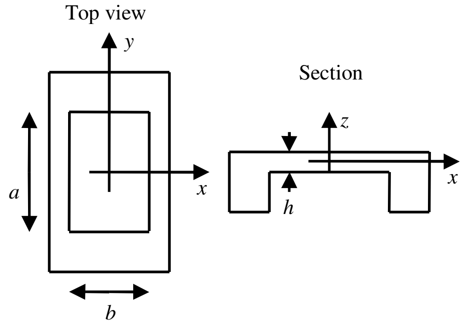

Fig.1: Geometry of the membrane.

The classical law describing the membrane bending made of orthotropic material under the pressure P for small deflections is [1]:

$$D_0 h^3 \Delta\Delta w(x,y) = P(x,y) \qquad (1)$$

where:

$$\Delta\Delta = \frac{\partial^4}{\partial x^4} + 2\alpha\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}, \quad D_0 = \frac{12E}{(1-\nu^2)},$$

D_0 is a rigidity, E is a Young modulus, ν is a Poisson ratio, α is an anisotropy coefficient and w is a change of membrane deflection due to applied pressure. In real devices, a residual stress σ_0 may remain in the material due to the technological process, changing the membrane stiffness. Then, the equation (1) takes the following form [1]:

$$D_0 h^3 \Delta\Delta w(x,y) + \sigma_0 h \Delta w(x,y) = P(x,y) \qquad (2)$$

where:

$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, and w is the total deflection. It was proved [2] that membrane deflection has linear dependence on applied pressure. Thus, the equation (1) can be simplified into following form:

$$C_1 \frac{D_0 h^3}{b^4} w_{\max} + C_2 \frac{\sigma_0 h}{b^2} w_{\max} = P \tag{3}$$

where w_{max} is the maximal membrane deflecion in its centre due to applied pressure and C_1 and C_2 are coefficients dependent on the ratio b/a and for C_1 on anisotropy coefficient (material crystallographic direction). The values of these coefficients are presented in [4].

In case of large deflections the assumption that there is no deformation in the middle plane of the membrane, is not valid and additional stresses must be taken into account [1]. Thus, the membrane bending is described with following equation:

$$\begin{gathered} D_0 h^3 \Delta\Delta w(x,y) + \sigma_0 h \Delta w(x,y) = P(x,y) + \\ + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} - 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} \end{gathered} \tag{4}$$

where N_x, N_y, N_{xy} are components of stress induced in the middle plane of the membrane. This makes the solution more complex as it has to be iterative. In each step the new value of induced stresses are obtained that is dependent on membrane deflection. The calculations stops when the value of deflection become constant. In general, the solution requires over a dozen of steps (depending on assumed precision) increasing the simulation time significantly. The advantages of simplified model describing only the maximal membrane deflection are even more visible in case of large deflections. It was found that the membrane bending is described with equation of third order as shown below:

$$\begin{gathered} C_1 \frac{D_0 h^3}{b^4} w_{\max} + C_2 \frac{\sigma_0 h}{b^2} w_{\max} + \\ + C_3 \frac{Eh}{(1-\nu) b^4} w_{\max}^3 = P \end{gathered} \tag{5}$$

where C_3 is a coefficient dependent on the ratio b/a and Poisson coefficient. In our previous work [4], the values of coefficients C_1 and C_2 have been estimated. In next paragraph, values of coefficient C_3 will be estimated including the same estimation procedure.

This analytical model allows also calculating the membrane deflection in one step as general formula for roots of third order equation exists. If we use Cardano's method using the normalized notation of equation of third order:

$$ax^3 + bx^2 + cx + d = 0 \tag{6}$$

for equation (5) we will find that coefficient h is always positive (except buckling state). Thus, only one stable real root exist:

$$x_1 = \sqrt[3]{-\frac{g}{2} + \sqrt{h}} + \sqrt[3]{-\frac{g}{2} - \sqrt{h}} - \frac{b}{3a} \tag{7}$$

where: $h = \frac{g^2}{4} + \frac{f^3}{27}$, $f = \frac{c}{a}$, $g = \frac{d}{a}$ for b=0.

III. Estimation procedure and results

In general three methods exist to find the coefficients of equation (5): analytical, experimental and using FEM. The first one needs to find exact solution of membrane deflection that is very complex and precision of this method is rather low [5]. The second one requires real structures that are exposed to influence of technological process. The third one requires just several simulations of test structure and interpolation process. As the FEM simulation is very precise as it uses the differential equation and one can simulate the exactly assumed structure, we decided to use this method, the same as in previous work [4].

In this paper we consider three kinds of materials: silicon cut in <100> and <110> direction (any other anisotropic material has to be considered separately) and isotropic material. The simulations are performed for membranes of ratio r in range of 1 to 3. Membranes with ratio greater than 3 are known as long rectangular meaning that the deflection become independent on ratio r [5]. The range of applied pressure is in range to cover the range of small and large deflection.

The method of estimating coefficient C_3 uses the transformation of equation (5) into the form:

$$A + B w_{\max}^2 = \frac{P}{w_{\max}} \tag{8}$$

Then, the function $\frac{P}{w_{\max}} = f(w_{\max}^2)$ should be a straight line.

The coefficient A of equation (8) is fixed using values from [4] and B is approximated. The exemplary graph of the function with its approximation is presented below:

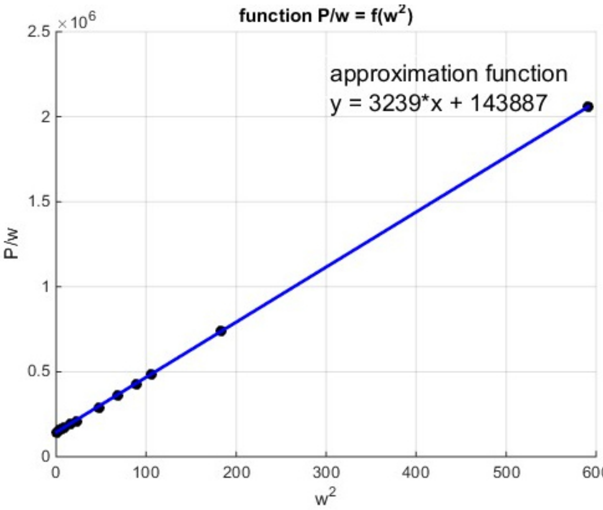

Fig.1: Exemplary function describing membrane deflection.

Then the coefficient C_3 is extracted from coefficient B. Because this coefficient depends on Poisson's ratio, in case of isotropic material further approximations has to be done to find the exact formula. The dependence of coefficient C3 on Poisson's ratio is presented in the figure below:

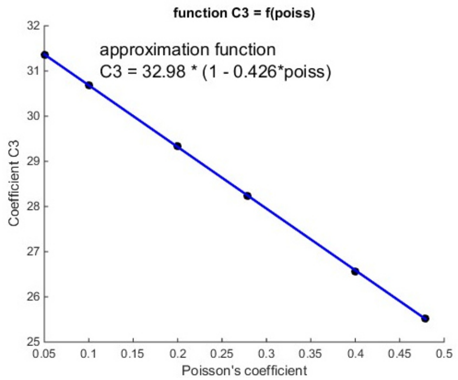

Fig.2: Dependence of coefficient C_3 on Poisson's ratio.

It has been found that this dependence can be describes with following relation:

$$C_3 = C_{3a}\left(1 - C_{3b} \cdot \nu\right) \quad (9)$$

where C_{3a} and C_{3b} are coefficients to be approximated.

The results of simulations are presented in the tables below wherein all approximated values of coefficients of equation (5) are presented:

TABLE 1

VALUES OF COEFFICIENTS C_1 AND C_2

Coefficient		C_1			C_2
Material		Isotropic	Silicon <100>	Silicon <110>	-
b/*a*	1	792.449	866.36	728.75	15.35
	1.5	455.894	488.32	427.76	11.42
	2	394.552	410.58	380.48	10.34
	2.5	382.591	389.44	376.51	9.88
	3	380.325	381.98	378.35	9.55

TABLE 2

VALUES OF COEFFICIENT C_3

Coefficient		C_3		
Material		Isotropic	Silicon <100>	Silicon <110>
b/*a*	1	32.07·(1-0.426ν)	29.12	34.91
	1.5	20.57·(1-0.426ν)	22.11	21.98
	2	19.11·(1-0.426ν)	21.39	21.19
	2.5	18.98·(1-0.426ν)	21.52	21.25
	3	18.85·(1-0.426ν)	21.55	21.23

Summarizing, the equation (5) combined with coefficients from tables 1 and 2 makes up the complete model of membrane bending including small and large deflection. It allows calculating maximal membrane deflection in its center of rectangular shape of any ratio *a*/*b*. The model is valid for any kind of residual stress (tensile and compressive) except buckling effect. The precision is very high as the model uses the data from FEM simulations. The only source of errors are numerical errors resulting from finite mesh density and rounding. This leads to resulting points that are not exactly on the straight line producing some errors in approximation process. Nevertheless, the difference should not be greater than 1%.

IV. CONCLUSIONS

In this paper, the analytical models of membrane bending were described including large deflections. The simplest model needs a specific values of coefficients to be found depending on material which the membrane is made of. The FEM simulation was used to estimate these values. This method provided obtaining the model that is very precise, comparable to FEM simulation. Moreover, this model is much faster than FEM simulation. Thus, it can be very useful in design phase wherein estimation of the device performance is desirable. Moreover, it can be also used in optimization phase reducing time consumption and simplifying it. Thus, it can be concluded, that the analytical model is a powerful tool that can be used as an supplement of FEM simulator.

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